Terminal singularities and U(1) factors in F-theory

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Based on: 22nn.nnnnn with A. Grassi, N. Raghuram, T. Weigand
2110.10159 with N. Raghuram

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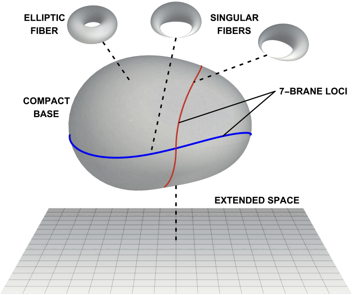
- Goal: investigate what happens when we introduce abelian gauge factors to F-theory models with non-crepant singularities
- Brief advertisement of recent proposal of heuristic method to read off $\mathrm{U}(1)$ charges without carrying out resolution, similar to Katz–Vafa

F-theory overview

- Elliptically fibered Calabi–Yau n-fold X:
  - Torus over each point in base B, $\pi\colon X \to B$
  - Has a section, $\sigma\colon B \to X$ s.t. $\pi \circ \sigma = \mathrm{Id}_B$
  - Complex structure τ encodes Type IIB axiodilaton



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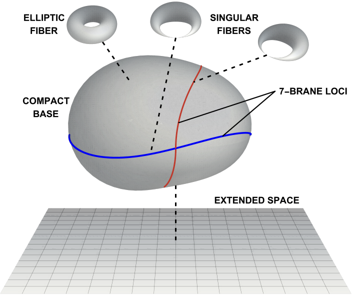

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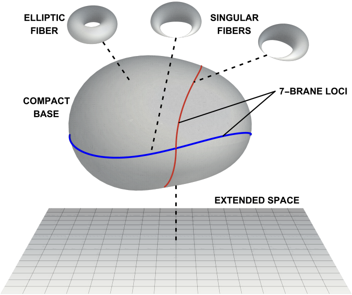

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- Gauge algebras and matter:
  - Codimension-one singularities (7-branes) $\longleftrightarrow$ nonabelian gauge algebras
  - Additional rational sections $\longleftrightarrow$ $\mathfrak{u}(1)$ gauge algebras
  - Codimension-two singularities $\longleftrightarrow$ massless matter

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- $K_{\tilde{X}} = K_X + \sum_i a_i E_i, \quad a_i$ "discrepancies"
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  - $a_i > 0$: terminal singularity
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- F-theory: resolution of singular point in fiber introduces $\operatorname{rk}\mathfrak{g}$ exceptional $\mathbb{P}^1$s
  - Gauge fields: $C_3 = A^i \wedge [E_i]$ and wrapped M2-branes
  - Localized matter: M2-branes wrapped on additional fibral curves C at codimension two
  - Mass along dual M-theory Coulomb branch: $m_0 \sim \mathrm{vol}(C)$

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- Number of uncharged massless hypermultiplets counted by Milnor number m_P:

$$\mathrm{CxDef}(X_3) = \underbrace{\left(\frac{1}{2}b_3(X_3) - 1 - \frac{1}{2}\sum_P m_P\right)}_{\text{def. independent of sing.}} + \underbrace{\sum_P m_P}_{\text{def. of sing.}}$$

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- $\mathfrak{sp}(k)$ $\mathrm{I}_{2k+1}^{\text{nonsplit}} \to \mathrm{I}_{2k+2}$ family [Grassi, Weigand '18]

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- Gauge algebra $\mathfrak{g} = \mathfrak{su}(2)$

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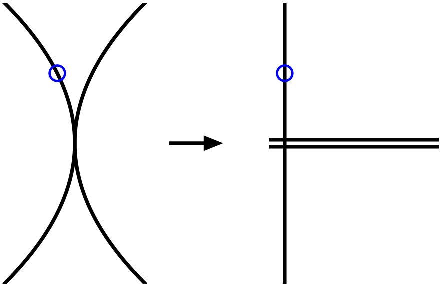

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- $\mathbb{Q}$-factorial terminal I_0^* at $\{\sigma = \tilde{c}_{1,1} = 0\}$ supporting $2 \times \mathbf{2}_0 + \mathbf{1}_0$
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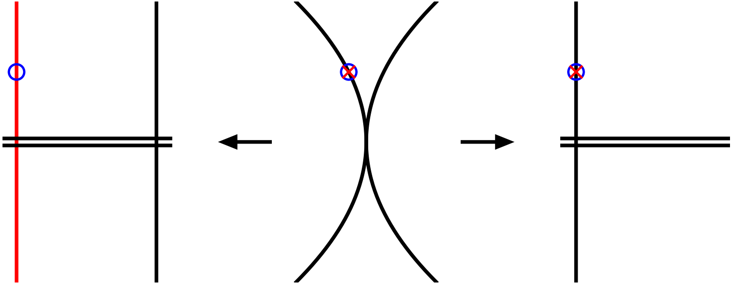

Reading off U(1) charges [Raghuram, APT '21]

Consider a model with gauge algebra $\mathfrak{g} \oplus \mathfrak{u}(1)$, with $\mathfrak{g}$ simply laced

- Codimension-one: singularity types G_i at loci $\{\sigma_i = 0\}$
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Codimension-one orders of vanishing:

$$\mathrm{ord}_1(\hat{z}) = 0\,, \quad \big(\mathrm{ord}_1(\hat{x}), \mathrm{ord}_1(\hat{y}), \mathrm{ord}_1(3\hat{x}^2 + f\hat{z}^4)\big) = \vec{\tau}_G(\nu)$$

Codimension-two orders of vanishing:

$$\mathrm{ord}_2(\hat{z}) = \frac{1}{2}\left(\frac{\prod_i d_{G_i}}{d_H} q^2 + \sum_i T_{G_i}(\nu_i) - T_H(\mu)\right)$$

$$\big(\mathrm{ord}_2(\hat{x}), \mathrm{ord}_2(\hat{y}), \mathrm{ord}_2(3\hat{x}^2 + f\hat{z}^4)\big) = (2,3,4) \times \mathrm{ord}_2(\hat{z}) + \vec{\tau}_H(\mu)$$

q: $\mathfrak{u}(1)$ charge
$[\hat{x} : \hat{y} : \hat{z}]$: section components
ν_i, μ: Integers denoting the components of the resolved fiber hit by the generating section
$\vec{\tau}_G(\nu)$: Triplet of integers dependent on G and ν
d_G: Number of elements in the center of G
$T_G(\nu) = (C_G^{-1})_{\nu\nu}$: νth diagonal entry of inverse Cartan matrix of G

Conclusions

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- The introduction of a $\mathrm{U}(1)$ factor to a model with $\mathbb{Q}$-factorial terminal singularities can give charge to previously uncharged localized massless hypermultiplets, changing the nature of the singularity to allow for a crepant resolution
- Can this be interpreted as an "enhancement" of discrete symmetries (possibly $\mathbb{Z}_1$) under which the localized multiplets are charged?
- What about models that are forced to give nonzero $\mathrm{U}(1)$ charge to all nonabelian-charged hypermultiplets?

Euler characteristic

$$\frac{1}{2}\chi_{\mathsf{top}} = \mathrm{KaDef}(X) - \mathrm{CxDef}(X) + \frac{1}{2}m_P$$

Compute:

- Resolve all codimension-one singularities
- Sum over all fibers:

$$\chi_{\mathsf{top}}(X_3) = \sum_i \chi_{\mathsf{top}}(X_{P_i}) \cdot B_i + \chi_{\mathsf{top}}(X_{\Sigma_1}) \cdot (2 - 2g(\Sigma_1) - \sum_i B_i)$$
$$+ 2 - 2g(\Sigma_0) + 3C + \sum_i \epsilon_i B_i$$

with

$$C = (-4K_B - \mathrm{ord}(f))(-6K_B - \mathrm{ord}(g)) - \sum_i \mu_{P_i}(f,g)B_i$$